

# The shortest path between two strings in product spaces

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## Abstract

We consider two independent and stationary measures over  $\chi^{\mathbb{N}}$ , where  $\chi$  finite or countable alphabet. For each pair of  $n$ -strings in the product space we define  $T_n^{(2)}$  as the length of the shortest path connecting one string to the other where the paths are generated by the underlying dynamics of the measure. For ergodic measures with positive entropy we prove that, for almost every pair of realizations  $(\mathbf{x}, \mathbf{y})$ ,  $T_n^{(2)}/n$  concentrates in one, as  $n$  diverges. Under mild extra conditions we prove a large deviation principle. This principle is linked to a quantity that compute the similarity between the two measures that we also introduce. We further prove its existence and other properties. We also show that the fluctuations of  $T_n^{(2)}$  converge (only) in distribution to a non-degenerated distribution. Several examples are provided for all results.

**Running head:** The shortest path between two strings.

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# 1 Introduction

Suppose one has to build a communication net consisting in nodes and links between nodes. A question of major interest is how to design the net such that it is easy to communicate from one node to another but without paying the cost of constructing a large number of links.

In this paper we study a quantity which describes the structural complexity of the net. Given two nodes, it gives the length of the shortest path from one node to another one. We consider the case where the nodes are given by the partition in  $n$ -cylinders or  $n$ -strings of the phase space: Specifically we consider a finite or countable set  $\chi$ . For each  $n \in \mathbb{N}$ , the nodes corresponds to the partition of  $n$ -cylinders or  $n$ -strings of  $\chi^{\mathbb{N}}$ . We consider also two independent probability measures over  $\Omega = \chi^{\mathbb{N}}$ . The address node is chosen according to a measure  $\mu$  and the source node according to a measure  $\nu$ . We assume that both measures are ergodic and that  $\mu$  is absolutely continuous with respect to  $\nu$ , otherwise the communication could be impossible. We denote with  $T_n^{(2)}$  the function that gives the length of the shortest path that communicates two  $n$ -strings. The link between this two strings is driven by the shift operator  $\sigma$  over  $\Omega$ . That is for  $\mathbf{x} = (x_0, x_1, \dots) \in \Omega$  one gets  $\sigma\mathbf{x} = (x_1, x_2, \dots)$ .

Let us introduce the cornerstone for this paper. It is the quantity that gives the minimum number of steps to get from a string to another one, and will be given nextly.

**Definition 1.1.** *The shortest-path function is defined by*

$$T_n^{(2)}(\mathbf{x}, \mathbf{y}) = \inf\{k \geq 1 : y_0^{n-1} \cap \sigma^{-k}(x_0^{n-1}) \neq \emptyset\}.$$

Here and ever after we write  $x_m^n$  as shorthand of  $x_m x_{m+1} \dots x_n$  for any  $0 \leq m \leq n \leq \infty$ . The random variable  $T_n^{(2)}$  is a two-dimensional version of the *shortest return function*  $T_n(\mathbf{x}) = \inf\{k \geq 1 \mid x_0^{n-1} \cap \sigma^{-k}(x_0^{n-1})\}$ . That is, it gives the length of the shortest path starting from and arriving to the same node. Its concentration phenomena have been already studied in [18, 5]. A large deviation principle was related to the Rényi entropy in [4, 9, 1]. Limiting theorems for its fluctuations were presented in [2, 3]. Since  $T_n$  considers starting and target sets being the same set,  $T_n$  and  $T_n^{(2)}$  have different nature. In topological terms:  $T_n$  describes a local, while  $T_n^{(2)}$  describes a global characteristic of the connection net.

In this paper we prove three fundamental theorems which describes the net through the statistical properties of  $T_n^{(2)}$ : Concentration, large deviations and fluctuations.

Firstly we prove that  $T_n^{(2)}/n$  converges almost surely around one. Our result holds when  $\mu$  has positive entropy and  $\nu$  verifies some specification property which prevents the net to be extremely sparse.

The concentration of  $T_n^{(2)}/n$  leads us to study its large deviation properties. Namely, the decaying to zero rate of the probability of deviating from one. We compute this rate under the additional condition that the measures verify

certain regularity condition. A similar condition was introduced and already related to the existence of a large deviation principle for the shortest return function  $T_n$  in [1].

The limiting rate of the large deviation function of  $T_n^{(2)}$  is determined by a quantity that deserves attention on its own. It gives a measure of similarity (or difference) between two measures (see definition 2.1). It is the expectation of the marginal distribution of order  $k$  of one of them with respect to the other. Since it is symmetric, they role are exchangeable in this definition. We call it the divergence of order  $k$ . We also study some of its properties that are used later on in the large deviation principle for  $T_n^{(2)}$  above mentioned. We provide several examples. In many cases the divergence results on an exponentially decreasing sequence on  $k$  and this leads to consider its limiting rate. One of our main results establishes the existence of the limiting rate function which is far for being evident. We use a kind of sub-additivity property but with a telescopic technique rather the classical linear one. We show that in particular, when the two measures coincide, this limit corresponds to the Rényi entropy of the measure at argument  $\beta = 2$  (see item (e) of examples (2.1)).

To describe the complexity of the net, we study de distribution of the shortest path function. We compute the distribution of a re-scaled version of  $T_n^{(2)}$  (namely,  $n - T_n^{(2)}$ ) and prove that it converges to a non-degenerated distribution which depends on the stationary measures  $\mu$  and  $\nu$ . The limiting distribution may depends on an infinite number of parameters if the measures do. This limiting distribution also depends on the divergence between the measures. As an application of this theorem we compute the proportion of pairs of  $n$ -strings which do not overlap (wich we call the *avoiding pairs* set).

In terms of the distribution of  $T_n^{(2)}$  we are not aware of any work which consider its behaviour in the context of stationary measures. There are some works which consider models of random graphs and present empirical data which adjust the distribution of the shortest path to Weibull or Gamma distributions [6, 23]. But even for classical models, for instance Erdős-Rényi graph, its full distribution has never been considered in the literature [7, 11, 21].

Since the random variables  $T_n^{(2)}$  are defined on the same probability space, we further ask about a stronger convergence. Our last result shows that  $n - T_n^{(2)}$  even do not converge in probability, and a lower bound for the distance between two consecutive terms of the sequence  $n - T_n^{(2)}$  is given.

Finally we think it is important also to highlight the connection of the shortest path function with the study of the Poincaré recurrence statistics. The waiting time function introduced by Wyner and Ziv in [26] is a well-studied quantity in the literature. Given two realizations  $\mathbf{x}, \mathbf{y} \in \chi^{\mathbb{N}}$ , it is the time expected until  $x_0^{n-1}$  appears in the realization  $\mathbf{y}$  of another process. That is

$$W_n(\mathbf{x}, \mathbf{y}) = \inf\{k \geq 1 : y_k^{k+n-1} = x_0^{n-1}\} .$$

Now, we have that the shortest path function is the minimum of the waiting times of  $x_0^{n-1}$ , taking the minimum over all the realizations  $\mathbf{z} \in \Omega$  that begin

with  $y_0^{n-1}$ . That is

$$T_n^{(2)}(\mathbf{x}, \mathbf{y}) = \inf_{\mathbf{z}: z_0^{n-1} = y_0^{n-1}} W_n(\mathbf{x}, \mathbf{z}) ,$$

A number of classical results are known for  $W_n$ . When both strings are chosen with the same measure, Shields showed that for stationary ergodic Markov chains  $\ln W_n/n \rightarrow h$  for almost every pair of realizations, as  $n$  diverges and  $h$  is the Shannon entropy of the measure [17]. Nobel and Wyner [14] further proved this result yet to convergence in probability for  $\alpha$ -mixing processes with a certain rate function  $\alpha$ . Marlon and Shields extended it to weak Bernoulli processes [13]. Yet, Shields [17] constructed an example of a very weak Bernoulli process in which the limit does not hold. Finally Wyner [25] proved that the distribution of  $W_n(\mathbf{x})\mu(\mathbf{x})$  converges to the exponential law for  $\psi$ -mixing measures. When both strings are chosen with possibly different measures, and the second one is a Markov chain, Kontoyiannis [12] showed that  $\ln W_n/n \rightarrow h(\mu) + h(\mu||\nu)$  for  $h(\mu||\nu)$  the relative entropy of  $\mu$  with respect to  $\nu$ .

This paper is organized as follows. In section 2 we introduce the divergence concept. Properties, examples and the proof of its existence are also included in this section. In section 3 we prove the concentration phenomena of the shortest path function. A large deviation principle is proved in section 4. The convergence of the shortest path distribution is presented in section 5. An application to calculate the self-avoiding pairs of strings appears also here. The non convergence in probability is shown in section 6.

## 2 The divergence between two measures

In this section, we define a central quantity for this paper. It gives, for two probability measures  $\mu$  and  $\nu$  over the same space  $(\Omega, \mathcal{F})$ , a degree of similarity between them.

**Definition 2.1.** *The  $k$ -divergence between  $\mu$  and  $\nu$  is defined by:*

$$\mathbb{E}_{\mu, \nu}(k) = \sum_{\omega \in \chi^k} \mu\nu(\omega) ,$$

(here and ever after, by  $\mu\nu(\omega)$  we mean  $\mu(\omega)\nu(\omega)$ ).

Let  $\mu_k$  ( $\nu_k$ ) be the projection of  $\mu$  ( $\nu$ ) over the first  $k$  coordinates of the space. The  $k$ -divergence is the mean of  $\mu_k$  with respect to  $\nu_k$ , or vice-versa. It is also the inner product of the  $|\chi|^k$ -vectors with entries given by the probabilities  $\mu(\omega)$  and  $\nu(\omega)$ ,  $\omega \in \chi^k$  (in any arbitrary ordering of the strings  $\omega$ ). Notice that  $\mathbb{E}_{\mu, \nu}(k)$  is symmetric ( $\mathbb{E}_{\mu, \nu}(k) = \mathbb{E}_{\nu, \mu}(k)$ ), and that it is not null if, and only if, the support of the two measures have non-empty intersection.

The next result says that the  $k$ -divergence does not decrease if we open a gap in the strings  $\omega \in \chi^k$ . As a corollary, we conclude that the divergence is not increasing in  $k$ . For simplicity, hereafter for  $\omega \in \chi^n$  and  $\xi \in \chi^m$  we denote by  $\omega\xi$  the  $n+m$ -string constructed by concatenation of  $\omega$  and  $\xi$ , formally  $\omega \cap \sigma^{-n}\xi$ .

**Lemma 2.1.** *Let  $i + g + j = k$  be non-negative integers. Then*

$$\mathbb{E}_{\mu,\nu}(k) \leq \sum_{\omega \in \chi^i; \zeta \in \chi^j} \mu\nu(\omega \cap \sigma^{-(i+g)}\zeta) . \quad (1)$$

*Proof.* Consider  $\omega \in \chi^i$ ;  $\xi \in \chi^g$ ;  $\zeta \in \chi^j$ . Let us write the cylinder  $\omega\sigma^{-i}\xi\sigma^{-(i+g)}\zeta \in \mathcal{F}_0^{k-1}$  by concatenating the three cylinders above. By removing the string  $\xi$  in  $\mu(\omega\xi\zeta)$

$$\begin{aligned} \sum_{\xi \in \chi^g} \mu\nu(\omega\xi\zeta) &\leq \sum_{\xi \in \chi^g} \mu\left(\omega \cap \sigma^{-(i+g)}\zeta\right) \nu(\omega\xi\zeta) \\ &= \mu\nu\left(\omega \cap \sigma^{-(i+g)}\zeta\right) . \end{aligned}$$

Summing over  $\omega$  and  $\zeta$  in the last display we get (1).  $\square$

As a direct consequence of the above proposition, we get that the  $k$ -divergence is monotonic.

**Corollary 2.1.** *If  $k < l$ , then  $\mathbb{E}_{\mu,\nu}(k) \geq \mathbb{E}_{\mu,\nu}(l)$ .*

*Proof.* This follows by taking  $i = k, g = 0, j = l - k$ .  $\square$

The above corollary proves that the  $k$ -divergence is a non-increasing function in  $k$ . In many cases, it decreases at an exponential rate. It is natural to ask about the existence of the limiting rate function, that is

$$\underline{\mathcal{R}} = \liminf_{k \rightarrow \infty} \left( -\frac{1}{k} \log \mathbb{E}_{\mu,\nu}(k) \right) ; \quad \overline{\mathcal{R}} = \limsup_{k \rightarrow \infty} \left( -\frac{1}{k} \log \mathbb{E}_{\mu,\nu}(k) \right) .$$

If both are equal, we denote it by  $\mathcal{R}$ <sup>1</sup>.

In what follows, we provide some examples and also a general condition in which the limiting rate exists. Further, in section 4.1, we will relate this limiting rate with a large deviation principle for  $T_n^{(2)}$ .

### Examples 2.1.

- (a) *Let  $\mu$  and  $\nu$  two independent measures with disjoint supports. Then  $\mathbb{E}_{\mu,\nu}(k) = 0$  for all  $k$ , and therefore  $\mathcal{R} = \infty$ .*
- (b) *Suppose that  $\mu$  and  $\nu$  concentrate their mass in a unique realization  $\mathbf{x}$  of the process. Then, for any  $\mathbf{y} \in \chi^{\mathbb{N}}$ ,  $\mu\nu(y_0^{k-1}) = 1$  if, and only if,  $y_0^{k-1} = x_0^{k-1}$  (and zero otherwise). Thus we get  $\mathcal{R} = 0$ .*
- (c) *If both measures  $\mu$  and  $\nu$  have independent and identically distributed marginals we get*

$$\mathbb{E}_{\mu,\nu}(k) = \sum_{\omega \in \chi^k} \mu\nu(\omega) = \left[ \sum_{x_0 \in \chi} \mu\nu(x_0) \right]^k = \mathbb{E}_{\mu,\nu}(1)^k .$$

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<sup>1</sup>Throughout this paper logarithms can be taken in any base.

Therefore, the limit  $\mathcal{R}$  exists and is given by

$$\mathcal{R} = -\log E_{\mu,\nu}(1) .$$

(d) Let  $\mu$  be a product of Bernoulli measures with parameter  $p$  and  $\nu$  a product of Bernoulli measures with parameter  $1-p$ . Then

$$\mathbb{E}_{\mu,\nu}(k) = \sum_{x_0^{k-1} \in \chi^k} \prod_{i=0}^{k-1} \mu\nu(x_i) = 2^k p^n (1-p)^k .$$

Then we get

$$\mathcal{R} = -\log[2p(1-p)] .$$

(e) If  $\mu = \nu$ , we get that  $\mathcal{R} = H_\mu(2)$ , where

$$H_\mu(\beta) = -\lim_{k \rightarrow \infty} \frac{1}{k(\beta-1)} \log \sum_{\omega \in \chi^k} \mu^\beta(\omega),$$

is the Rényi entropy of the measure  $\mu$  (provided that it exists).

(f) A case where the limiting rate does not exist. Let  $\chi = \{0, 1\}$ , and let  $\mu$  be a measure concentrated on the realization:

$$\mathbf{x} = 0^2 1^{2^2} 0^{2^3} 1^{2^4} \dots 0^{2^k} 1^{2^{k+1}} \dots$$

where  $a^j$  means the  $j$ -string  $aa \dots a \in \chi^j$ . On the other hand, let  $\nu$  be a product of Bernoulli measures with  $p \neq 1/2$ . By a direct computation, we get that  $\mathbb{E}_{\mu,\nu}(k) = \nu(x_0^{k-1})$ . Since the proportion of 0's and 1's in  $x_0^{k-1}$  does not converge as  $k$  goes to infinity, we get  $\mathcal{R} \neq \overline{\mathcal{R}}$ .

(g) Let  $\nu$  be an ergodic, positive entropy measure. By the Shannon-McMillan-Breiman Theorem,  $-1/k \log \nu(x_0^{k-1})$  converges to  $h(\nu)$ , for almost every  $\mathbf{x} \in \chi^\mathbb{N}$ , where  $h(\nu)$  is the entropy of  $\nu$ . Let  $\mathbf{x}$  be one of such sequences. Let  $\mu$  be a measure concentrated on  $\mathbf{x}$ . Then  $\mathbb{E}_{\mu,\nu}(k) = \nu(x_0^{k-1})$  and  $\mathcal{R} = h(\nu)$ .

The following theorem gives sufficient conditions for the existence of the limiting rate  $\mathcal{R}$ . Its proof uses a kind of sub-additive property. But here, instead of the classical linear iteration of the sub-additivity property, we use a geometric iteration. To prove the existence of the limiting rate function  $\mathcal{R}$  we use a kind of  $g$ -regular condition which is a version of the condition introduced in [1]. This condition was used to prove a large deviation principle for the shortest return function  $T_n$  of a string to itself. That principle related the deviations of  $T_n$  to the Rényi entropies of the measure. Examples which shows its generality and also properties can be also found there. Let  $g$  be a fixed non-negative integer. Define

$$\psi_{\mu,g}^+(i,j) = \sup_{\omega \in \chi^i, \xi \in \chi^j} \frac{\mu(\sigma^{-(i+g)}\xi \mid \omega)}{\mu(\xi)} .$$

**Theorem 2.1.** Let  $\psi_g^+ = \max\{\psi_{\mu,g}^+, \psi_{\nu,g}^+\}$ . Suppose there exist positive constants  $K > 0$  and  $\epsilon$  such that

$$\log \psi_g^+(i, j) \leq K \frac{i+j}{[\log(i+j)]^{1+\epsilon}}. \quad (2)$$

Then  $\mathcal{R}$  does exist.

*Proof.* Let us take  $\omega \in \chi^i, \xi^g, \zeta \in \chi^j$ . As in the proof of Lemma 2.1,  $\sum_{\xi \in \chi^g} \mu\nu(\omega\xi\zeta) \leq \mu\nu(\omega \cap \sigma^{-(i+g)}\zeta)$ . Further, by the  $\psi_g$ -regular condition

$$\mu(\omega \cap \sigma^{-(i+g)}\zeta) \leq \psi_g^+(i, j)\mu(\omega)\mu(\zeta).$$

And the same holds for  $\nu$ . Call  $f(k) = \log \mathbb{E}_{\mu,\nu}(k)$ . Also, call  $c_g(i, j) = 2 \log \psi_g^+(i, j)$ . Summing up in  $\omega$  and  $\zeta$  and taking logarithm, by the inequalities above, we conclude that for all  $i, j$ ,

$$\begin{aligned} f(i+g+j) &\leq \log(\psi_g^+(i, j)\psi_g^+(i, j)) + \log \sum_{\omega \in \chi^i} \mu\nu(\omega) + \log \sum_{\zeta \in \chi^j} \mu\nu(\zeta) \\ &= c_g(i, j) + f(i) + f(j). \end{aligned} \quad (3)$$

Now we use a kind of sub-additivity argument. Let  $(n_t)_{t \in \mathbb{N}}$  an increasing sequence of non-negative integers such that

$$\liminf_{n \rightarrow \infty} \frac{f(n)}{n} = \lim_{t \rightarrow \infty} \frac{f(n_t)}{n_t}. \quad (4)$$

Consider the sequence  $\tilde{n}_t = n_t + g$ , with  $t \in \mathbb{N}$ . Fix  $t$ . Firstly, for any positive integer  $n \geq \tilde{n}_t$  write  $n = \tilde{n}_t m + r$ , with positive integers  $m, r$  such that  $0 \leq r < \tilde{n}_t$ . Apply (3) with  $i = \tilde{n}_t m - g, j = r$ , and gap  $g$  to get

$$f(n) \leq c_g(\tilde{n}_t m - g, r) + f(\tilde{n}_t m - g) + f(r). \quad (5)$$

Now, we write  $m$  in base 2. For this, there exist a positive integer  $\ell(m)$  and non-negative integers  $\ell_1 < \ell_2 < \dots < \ell_{\ell(m)}$ , such that  $m = \sum_{s=1}^{\ell(m)} 2^{\ell_s}$ . Iterating (3) with  $i = \tilde{n}_t \sum_{s=1}^{u-1} 2^{\ell_s} - g$  and  $j = \tilde{n}_t 2^{\ell_u} - g$ , for  $u = 2, \dots, \ell(m)$ , we have that for the middle term in the right hand side of (5)

$$f(\tilde{n}_t m - g) \leq \sum_{u=2}^{\ell(m)} c_g \left( \tilde{n}_t \sum_{s=1}^{u-1} 2^{\ell_s} - g, \tilde{n}_t 2^{\ell_u} - g \right) + \sum_{u=1}^{\ell(m)} f(\tilde{n}_t 2^{\ell_u} - g). \quad (6)$$

The first sum in the righthand side is zero in case  $\ell(m) = 1$ . Finally, we decompose the argument in the last summation. For any  $n \in \mathbb{N}$  of the form  $n = \tilde{n}_t 2^\ell - g$  we apply (3) with  $i = j = \tilde{n}_t 2^{\ell-1} - g$ , to get

$$f(\tilde{n}_t 2^\ell - g) \leq c_g(\tilde{n}_t 2^{\ell-1} - g, \tilde{n}_t 2^{\ell-1} - g) + 2f(\tilde{n}_t 2^{\ell-1} - g).$$

An iteration of the above inequality leads to

$$f(\tilde{n}_t 2^\ell - g) \leq \sum_{s=0}^{\ell-1} 2^{\ell-s-1} c_g(\tilde{n}_t 2^s - g, \tilde{n}_t 2^s - g) + 2^\ell f(\tilde{n}_t - g). \quad (7)$$

Observe that  $\tilde{n}_t - g = n_t$ . Collecting (5), (6), (7) we conclude that the limit superior of  $f(n)/n$  is upper bounded by the limit superior of  $I + II + III + IV$  where

$$\begin{aligned} I &= \frac{c_g(\tilde{n}_t m - g, r)}{\tilde{n}_t m - g + r}, \\ II &= \frac{1}{\tilde{n}_t m} \sum_{u=2}^{\ell(m)} c_g \left( \tilde{n}_t \sum_{s=1}^{u-1} 2^{\ell_s} - g, \tilde{n}_t 2^{\ell_u} - g \right), \\ III &= \frac{1}{\tilde{n}_t m} \sum_{u=1}^{\ell(m)} \sum_{s=0}^{\ell_u-1} 2^{\ell_u-s-1} c_g(\tilde{n}_t 2^s - g, \tilde{n}_t 2^s - g), \\ IV &= \frac{f(r)}{\tilde{n}_t m} + \frac{\sum_{u=1}^{\ell(m)} 2^{\ell_u} f(n_t)}{\tilde{n}_t m}. \end{aligned}$$

As  $m$  diverges, the first term in  $IV$  vanishes since  $0 \leq r < \tilde{n}_t$ . The second one is bounded by  $f(n_t)/n_t$ .  $I$  goes to zero by (2). We recall that  $II = 0$  in case  $\ell(m) = 1$ . Otherwise we also use condition (2) to get the following upper bound

$$\frac{K}{m} \sum_{u=2}^{\ell(m)} \frac{\sum_{s=1}^u 2^{\ell_s}}{[\log(\tilde{n}_t \sum_{s=1}^u 2^{\ell_s} - 2g)]^{1+\epsilon}}.$$

The inner summation is trivially bounded by  $3 \leq \sum_{s=1}^u 2^{\ell_s} \leq 2^{\ell_u+1}$ . Since  $m = \sum_{u=1}^{\ell(m)} 2^{\ell_u}$ , it follows that  $II \leq 2K/[\log n_t]^{1+\epsilon}$ . Lastly using also (2) we get that  $III$  is upper bounded by

$$\frac{K}{m} \sum_{u=1}^{\ell(m)} 2^{\ell_u} \sum_{s=0}^{\ell_u-1} \frac{1}{[\log \tilde{n}_t 2^{s+1} - 2g]^{1+\epsilon}}.$$

Changing the constant  $K$  we can take here logarithm base 2. The argument in the logarithm is lower bounded by  $n_t 2^{s+1}$ . Now we use that the sum of a decreasing sequence is bounded above by its first term plus the integral definite by the first and last terms in the sum. Thus, the rightmost sum in the above display is bounded by

$$\sum_{s=1}^{\infty} \frac{1}{[s + \log n_t]^{1+\epsilon}} \leq \frac{1}{[1 + \log n_t]^{1+\epsilon}} + \frac{1}{\epsilon [1 + \log n_t]^{\epsilon}},$$

which goes to zero as  $t$  diverges. Summarizing we conclude that

$$\limsup_{n \rightarrow \infty} \frac{f(n)}{n} \leq \frac{f(n_t)}{n_t} + \frac{K}{[\log n_t]^{1+\epsilon}} + \frac{1}{\epsilon [1 + \log n_t]^{\epsilon}}.$$

The inequality holds for every  $t$ . If we take  $t \rightarrow \infty$  then we finishes the proof since  $f(n_t)/n_t$  converges by hypothesis.  $\square$



### 3 Concentration

The main result of this section says that  $T_n/n$  converges almost surely to one. The proof is divided in two parts. The first one proves that the limit inferior is lower-bounded by one and the second one that the limit superior is upper-bounded by one. For the last one we assume that the process verifies the very weak specification property (see def. (3.1) below).

There are several definition of specification, the first one introduced by Bowen[8], many follow him with some divergence in the nomenclature [19, 10], or in weaker forms (se for instance [20, 22, 24]). Basically they mean that, for any given set of strings, they can be observed (at least in one single realization of the process) with bounded gaps between them. Sometimes it is required the realization to be periodic. for simplicity to the reader, we present our condition here. It is easy to see that it is verified for a large class of stochastic processes and is less restrictive than the previous ones. Examples are provided below.

**Definition 3.1.**  $(\chi^{\mathbb{N}}, \mu, \sigma)$  is said to have the very weak specification property (VWSP) if there exists a function  $g : \mathbb{N} \rightarrow \mathbb{N}$  with  $\lim_{n \rightarrow \infty} g(n)/n = 0$ , that verifies the following: For any pair of strings  $\omega, \xi \in \chi^n$ , there exists a  $\mathbf{x} \in \chi^{\mathbb{N}}$  such that,

$$x_0^{n-1} = \omega \quad \text{and} \quad x_{n+g(n)}^{2n+g(n)-1} = \xi .$$

#### Examples 3.1.

- (a) Any process with complete grammar verifies definition 3.1 with  $g(n) = 0$ . We recall that a probability measure  $\mu$  defined over  $\chi^{\mathbb{N}}$  is said to have complete grammar if, for all  $n \in \mathbb{N}$  we get  $\mu(\omega) > 0$  for all  $\omega \in \chi^n$ .
- (b) An irreducible and aperiodic Markov chain over a finite alphabet  $\chi$  and stationary measure  $\mu$  verifies the VWSP with  $g(n) \leq |\chi|$ .
- (c) We first construct a renewal process  $(X_n)_{n \in \mathbb{N}}$  as an image of the House of Cards Markov chain  $(Y_n)_{n \in \mathbb{N}}$  with irreducible and aperiodic transition matrix  $Q$  given by

$$\begin{aligned} Q(y, 0) &= 1 - q_y, \\ Q(y, y+1) &= q_y, \end{aligned}$$

$y \in \{0, 1, 2, \dots\}$ . Figure 1 represents the transitions of this process.

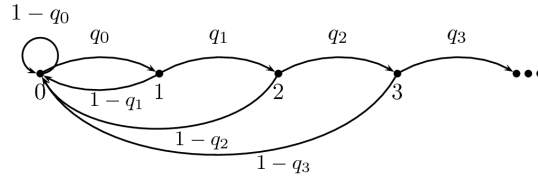


Figure 1: House of cards Markov chain  $(Y_n)_{n \geq 0}$

Let  $X_n = 1$  if  $Y_n = 0$ , and  $X_n = 0$  if  $Y_n \neq 0$ , indicating the "renewal" of  $(Y_n)_{n \geq 0}$ . Take  $q_y = 1$ , for all  $n^2 \leq y \leq n^2 + n$  for some  $n \in \mathbb{N}$ , and any other  $0 < q_y < 1$  for the remaining coefficients to warranty the Markov chain is positive recurrent. Obviously  $(X_n)_{n \geq 0}$  has not complete grammar. It is easy to see that  $g(n) \leq \sqrt{n}$  and that this bound is actually sharp, taking  $\omega = \xi = 01^{n^2-1} \in \chi^{n^2}$ . The stationary measure of the House of Cards Markov chain itself is an example that does not verify the VWSP.

Now we can state the main result of this section.

**Theorem 3.1.** *Suppose  $\mu$  has positive entropy, then*

$$(a) \liminf_{n \rightarrow \infty} \frac{T_n^{(2)}}{n} \geq 1, \quad \mu \times \nu - a.e.$$

(b) *In addition, if  $\nu$  verifies the VWSP, then*

$$\lim_{n \rightarrow \infty} \frac{T_n^{(2)}}{n} = 1, \quad \mu \times \nu - a.e.$$

Before proving the above result, let us introduce a family of sets and a result that will be useful for the proof.

**Definition 3.2.** *For each  $k \in \{1, \dots, n-1\}$  define the set of pairs  $(\mathbf{x}, \mathbf{y}) \in \chi^{\mathbb{N}} \times \chi^{\mathbb{N}}$  such that the firsts  $k$  symbols of  $x_0^{n-1}$  coincide exactly with the last  $k$  symbols of  $y_0^{n-1}$ . Namely*

$$R_n^{(2)}(k) = \{(\mathbf{x}, \mathbf{y}) \in \chi^{\mathbb{N}} \times \chi^{\mathbb{N}} : y_{n-k}^{n-1} = x_0^{k-1}\}.$$

**Lemma 3.1.** *For  $k < n$ , it holds*

$$\{T_n^{(2)} \leq k\} \subseteq \bigcup_{i=n-k}^{n-1} R_n^{(2)}(i).$$

*In addition, if  $\nu$  satisfies the complete grammar condition, then the equality holds.*

*Proof.* By definition,  $(\mathbf{x}, \mathbf{y})$  belongs to  $\{T_n^{(2)} \leq k\}$ , if and only if, there is  $\mathbf{z} \in \chi^{\mathbb{N}}$  and  $1 \leq i \leq k$  such that  $z_0^{n-1} = y_0^{n-1}$  and  $z_i^{i+n-1} = x_0^{n-1}$ . In particular, since  $n-1 \geq i$ , we have  $y_i^{n-1} = x_0^{n-i-1}$ , which in turns says that  $(\mathbf{x}, \mathbf{y}) \in R_n^{(2)}(n-i)$ . For the equality, notice that for any pair  $(\mathbf{x}, \mathbf{y}) \in R_n^{(2)}(i)$ ,  $i \in \{n-k, \dots, n-1\}$  we get that  $x_0^{i-1} = y_{n-i}^{n-1}$ . The complete grammar condition assures that there exists  $\mathbf{z} \in \chi^{\mathbb{N}}$  such that  $z_0^{n-1} = y_0^{n-1}$  and  $z_i^{i+n-1} = x_0^{n-1}$ . This concludes the proof.  $\square$

The next lemma gives the key connexion with the divergence of  $\mu$  and  $\nu$ .

**Lemma 3.2.** *For  $1 \leq k < n$ , it holds*

$$\mathbb{P}(R_n^{(2)}(k)) = \mathbb{E}_{\mu, \nu}(k).$$

*Proof.* If  $1 \leq k < n$ , then

$$\mathbb{P}(R_n^{(2)}(k)) = \mathbb{P}(x_0^{k-1} = y_{n-k}^{n-1}) = \sum_{\omega \in \chi^k} \mu\nu(\omega) .$$

Since the last term above is equal to  $\mathbb{E}_{\mu,\nu}(k)$ , we finish the proof.  $\square$

Now we are able to prove theorem 3.1.

*Proof of theorem 3.1.* For item (a), let  $h > 0$  the entropy of  $\mu$ . Since  $\mu$  is ergodic, the Shannon-McMillan-Breiman Theorem says that

$$-\lim_{n \rightarrow \infty} \frac{1}{n} \log \mu(x_0^{n-1}) = h ,$$

along  $\mu$ -almost every  $\mathbf{x} \in \chi^{\mathbb{N}}$ . By Egorov's Theorem, for every  $0 < \epsilon < h$ , there exists a subset  $\Omega_\epsilon$  of  $\Omega$ , where this convergence is uniform and  $\mu(\Omega_\epsilon) \geq 1 - \epsilon$ . That is, for all  $\epsilon > 0$ , there exists a  $k_0(\epsilon)$  such that for all  $k > k_0(\epsilon)$

$$e^{-k(h+\epsilon)} < \mu(x_0^{k-1}) < e^{-k(h-\epsilon)} , \quad (8)$$

for all  $\mathbf{x} \in \Omega_\epsilon$ . Making the product with  $\nu$ , and using lemmas 3.1 and 3.2 we get

$$\begin{aligned} \mathbb{P}\left(\{T_n^{(2)} \leq (1-\epsilon)n\} \cap \Omega_\epsilon \times \Omega\right) &\leq \sum_{j=\lfloor \epsilon n \rfloor}^{n-1} \mathbb{P}\left(R_n^{(2)}(j) \cap \Omega_\epsilon \times \Omega\right) \\ &= \sum_{j=\lfloor \epsilon n \rfloor}^{n-1} \sum_{\omega \in \chi^j \cap \Omega_\epsilon} \mu\nu(\omega) \\ &\leq \sum_{j=\lfloor \epsilon n \rfloor}^{n-1} e^{-j(h-\epsilon)} , \end{aligned}$$

where the last inequality was obtained using (8). A direct computation gives

$$\sum_{n=1}^{\infty} \mathbb{P}(T_n^{(2)} \leq (1-\epsilon)n) \leq \frac{1}{1 - e^{-(h-\epsilon)}} .$$

By Borel-Catelli's Lemma,  $\{T_n^{(2)} \leq (1-\epsilon)n\}$  occurs only finitely many times. We conclude

$$\liminf_{n \rightarrow \infty} \frac{T_n^{(2)}}{n} \geq 1 - \epsilon , \quad \mu \times \nu - \text{a.s. in } \Omega_\epsilon \times \Omega . \quad (9)$$

Since  $\epsilon$  is arbitrary, this finishes the proof of (a).

Now we prove item (b). Since definition 3.1 implies that  $T_n^{(2)} \leq n + g(n)$ , we divide both sides by  $n$  and get

$$\limsup_{n \rightarrow \infty} \frac{T_n^{(2)}}{n} \leq 1, \quad \mu \times \nu - \text{a.e.}$$

Combining this with (a), we finish the proof of item (b).  $\square$

## 4 Large deviations

In the previous section we showed that  $T_n^{(2)}/n$  concentrates its mass in 1, as  $n$  diverges. Here, we present the deviation rate for this limit. Since the VWSP implies that

$$\mathbb{P}\left(\frac{T_n^{(2)}}{n} > 1 + \epsilon\right) = 0, \quad \forall n > n_0(\epsilon),$$

it is only meaningful to consider the lower deviation.

**Definition 4.1.** We define the  $\liminf$  and  $\limsup$  for the lower deviation rate, respectively, as

$$\underline{\Delta}(\epsilon) = \liminf_{n \rightarrow \infty} \frac{1}{n} \left| \log \mathbb{P}\left(\frac{T_n^{(2)}}{n} < 1 - \epsilon\right) \right|,$$

and

$$\overline{\Delta}(\epsilon) = \limsup_{n \rightarrow \infty} \frac{1}{n} \left| \log \mathbb{P}\left(\frac{T_n^{(2)}}{n} < 1 - \epsilon\right) \right|.$$

If  $\underline{\Delta} = \overline{\Delta}$  we write simply  $\Delta$ .

We recall that the complete grammar condition assures that  $T_n^{(2)} \leq n$ .

**Theorem 4.1.** Let  $\mu$  and  $\nu$  two stationary probability measures defined over  $\chi^{\mathbb{N}}$ . Then

$$(a) \quad \underline{\Delta}(\epsilon) \leq \epsilon \underline{\mathcal{R}} \quad \text{and} \quad \overline{\Delta}(\epsilon) \leq \epsilon \overline{\mathcal{R}}.$$

(b) Suppose that  $\nu$  has complete grammar. Then the equalities hold in (a).

The  $\psi_g$ -regularity of the measure assures the existence of  $\mathcal{R}$ .

**Corollary 4.1.** Under conditions of theorem 4.1, suppose yet that  $\nu$  has complete grammar. Then  $\Delta(\epsilon) = \epsilon \mathcal{R}$ .

*Proof of theorem 4.1.* By lemma 3.1, we have

$$\left\{ \frac{T_n^{(2)}}{n} < 1 - \epsilon \right\} \subseteq \bigcup_{j=\lceil n\epsilon \rceil}^{n-1} R_n^{(2)}(j),$$

with equality if the process has complete grammar. In this case, considering just the first set in the union, we also have

$$R_n^{(2)}(\lceil n\epsilon \rceil) \subseteq \left\{ \frac{T_n^{(2)}}{n} < 1 - \epsilon \right\}.$$

Thus, by lemma 3.2

$$\mathbb{E}_{\mu,\nu}(\lceil n\epsilon \rceil) \leq \mathbb{P}\left(\frac{T_n^{(2)}}{n} < 1 - \epsilon\right) \leq \sum_{j=\lceil n\epsilon \rceil}^{n-1} \mathbb{E}_{\mu,\nu}(j) .$$

Now we take logarithm, divide by  $n$ , take limit and use that the divergence is non increasing. An exchange of variables ends the proof.  $\square$

## 5 Convergence in law

In this section we prove the convergence of the normalized distribution of  $T_n^{(2)}$  to a non-degenerated distribution. We also present several examples and provide an application for the main result of the section.

To state the result we need first to introduce the coefficients that appears in the theorem.

**Definition 5.1.** Set  $a_{n-1,n}^{(2)} = 0$  and for every  $1 \leq k \leq n-2$ , define:

$$\begin{aligned} \bullet \quad a_{k,n}^{(2)} &= \sum_{m=k+1}^{n-1} \sum_{\omega \notin \cup_{j=k}^{m-1} R_m(j)} \mu\nu(\omega) \quad . \\ \bullet \quad a_k^{(2)} &= \sum_{m=k+1}^{\infty} \sum_{\omega \notin \cup_{j=k}^{m-1} R_m(j)} \mu\nu(\omega) \quad . \end{aligned}$$

Here, we define by  $R_n(k)$  a set which is a one dimensional version of  $R_n^{(2)}(k)$ . Namely

$$R_n(k) = \{x_0^{n-1} \in \chi^n : x_{n-k}^{n-1} = x_0^{k-1}\} .$$

Now we can state the main result of this section.

**Theorem 5.1.** Suppose  $\nu$  has complete grammar. Then, for all  $1 \leq k \leq n-1$ , it holds:

$$\begin{aligned} (a) \quad \mathbb{P}(n - T_n^{(2)} \geq k) &= \mathbb{E}_{\mu,\nu}(k) + a_{k,n}^{(2)} \quad . \\ (b) \quad \lim_{n \rightarrow \infty} \mathbb{P}(n - T_n^{(2)} \geq k) &= \mathbb{E}_{\mu,\nu}(k) + a_k^{(2)} \quad . \end{aligned}$$

In the next examples we discuss several cases of applications of the above theorem.

### Examples 5.1.

(a) The theorem does not warrant that the limiting object  $(\mathbb{E}_{\mu,\nu}(k) + a_k^{(2)})_{n \in \mathbb{N}}$  actually defines a distribution law. For instance, take  $\mu$  concentrated on the unique sequence  $\mathbf{x} = (11111 \dots)$ . Let  $\nu_p$  be a product of Bernoulli measure with success probability  $p$ . Let  $0 < \lambda < 1$ , and define  $\nu = \lambda\mu + (1-\lambda)\nu_p$ . Clearly,  $\mu$  is absolutely continuous with respect to  $\nu$ , which has complete grammar. It is easy to compute  $\mathbb{P}(T_n^{(2)} = 1) = \nu(1^n) = \lambda + (1-\lambda)p^n$ . Thus  $\mathbb{P}(n - T_n^{(2)} = \infty) \geq \lambda$  and  $n - T_n^{(2)}$  does not converge to a limiting distribution.

(b) Under mild conditions one gets that the limiting object is actually a distribution. To that, it is enough to give conditions in which  $\mathbb{P}(n - T_n^{(2)} = \infty) = \lim_{k \rightarrow \infty} \mathbb{E}_{\mu, \nu}(k) + a_k^{(2)} = 0$ . Directly from its definition  $a_k^{(2)} \leq \sum_{j=k+1}^{\infty} \mathbb{E}_{\mu, \nu}(j)$ . Thus, the limiting function defines a distribution if  $\sum_{j=k}^{\infty} \mathbb{E}_{\mu, \nu}(j)$  goes to zero as  $k$  diverges. It holds if  $\min\{\max_{\omega \in \chi^n} \mu(\omega), \max_{\omega \in \chi^n} \nu(\omega)\}$  is summable. Notice that this is not the case in the example above.

(c) When  $\mu = \nu$ , we recover in the limit, the same limit distribution of the re-scaled shortest return function  $n - T_n$ . In particular, if  $\mu$  is a product measure, we recover the limit distribution obtained in [2].

(d) The following example shows a process that has complete grammar, and then  $n - T_n^{(2)}$  converges. On the contrary, the re-scaled shortest return function  $n - T_n$  does not converge as shown in [3]. This is due to the fact that the process is not  $\beta$ -mixing. The process  $(X_n)_{n \geq 0}$  is defined over  $\chi = \{0, 1\}$  in the following way. Let  $X_0$  be uniformly chosen over  $\{0, 1\}$  and independent of everything. The remaining variables are conditionally independent given  $X_0$  and defined by

$$\mu(X_{2n} = X_0) = 1 - \epsilon = 1 - \mu(X_{2n-1} = X_0).$$

It is obvious that the process has complete grammar, has a unique invariant measure with marginal distribution of  $X_n, n \geq 1$  being the uniform one. Take now  $\nu = \mu$ . So, they verify the hypothesis of theorem 5.1 and therefore  $n - T_n^{(2)}$  converges.

## Application

We call  $\{(x_0^{n-1}, y_0^{n-1}) \in \chi^n \times \chi^n \mid T_n^{(2)}(\mathbf{x}, \mathbf{y}) = n\}$  the set of *avoiding pairs*, since only in this case the chosen two strings do not overlap. As far as we know it was never considered in the literature. A similar quantity was actually considered, the set of *self-avoiding strings*, which is defined similarly, using the shortest return function  $T_n$ , Namely  $\{T_n = n\}$ . It is read as the set of strings which do not over itself. It was first studied in [16], when the authors treated a problem related to Cellular Automata. They considered only the case of a uniform product measure. Using an argument due to S. Janson, the authors calculate the proportion of self-avoiding strings of length  $n$ . This result was generalized in [2], and posteriorly in [3] to independent and  $\beta$ -mixing processes respectively.

The next result follows immediately from theorem 5.1, and gives us the probability of the set of avoiding pairs.

**Corollary 5.1.** *Under the conditions of theorem 5.1, the measure of the avoiding pairs set is given by*

- (a)  $\mathbb{P}(T_n^{(2)} = n) = 1 - \mathbb{E}_{\mu, \nu}(1) - a_{1,n}^{(2)}.$
- (b)  $\lim_{n \rightarrow \infty} \mathbb{P}(T_n^{(2)} = n) = 1 - \mathbb{E}_{\mu, \nu}(1) - a_1^{(2)}.$

*Proof of Theorem 5.1.* The main idea of the proof is to bring the two-dimensional problem to a one-dimensional one. By lemma 3.1 and since  $\nu$  has complete grammar, we get that

$$\left\{n - T_n^{(2)} \geq k\right\} = \bigcup_{j=k}^{n-1} R_n^{(2)}(j) .$$

Decompose the right-hand side of the above equality in disjoint sets to get

$$R_n^{(2)}(k) \cup \bigcup_{j=k+1}^{n-1} R_n^{(2)}(j) \setminus \bigcup_{l=k}^{m-1} R_n^{(2)}(l) . \quad (10)$$

By lemma 3.2,  $\mathbb{P}(R_n^{(2)}(k)) = \mathbb{E}_{\mu, \nu}(k)$ . For the second set in (10),  $(\mathbf{x}, \mathbf{y}) \in R_n^{(2)}(j)$ , if and only if  $x_0^{j-1} = y_{n-j}^{n-1}$ . Further,  $(\mathbf{x}, \mathbf{y})$  does not belong to  $\bigcup_{l=k}^{m-1} R_n^{(2)}(l)$  if, and only if  $x_0^{l-1} \neq y_{n-l}^{n-1}$ , for all  $k \leq l \leq m-1$ . Since this last two conditions depend only on the values of  $x_0, \dots, x_{m-1}, y_{n-m}, \dots, y_{n-1}$  we get that the probability of the rightmost set in (10) equals to

$$\sum_{j=k+1}^{n-1} \sum_{\omega \notin \bigcup_{l=k}^{m-1} R_m(l)} \mu\nu(\omega) = a_{k,n}^{(2)} . \quad (11)$$

Since  $\mathbb{E}_{\mu, \nu}(k)$  does not depend on  $n$ , the limit of the probability of the left-side set in (10) when  $n$  goes to infinity only depends on its second term, which is a non-decreasing function on  $n$ . Each term is also bounded above by 1. Therefore it converges, and this concludes the proof.  $\square$

## 6 Non-convergence in probability

In the present section we show that the convergence of  $n - T_n^{(2)}$  cannot be stronger than convergence in distribution. The result is stated as follows.

**Proposition 6.1.** *Under the conditions of theorem 5.1,  $n - T_n^{(2)}$  does not converge in probability.*

*Proof.* It is sufficient to show that

$$\mathbb{P}(|n+1 - T_{n+1}^{(2)} - (n - T_n^{(2)})| > \epsilon),$$

does not converge to zero. Take  $0 < \epsilon < 1$ . It is obvious that

$$\{T_{n+1}^{(2)} = T_n^{(2)}\} \subset \{|n+1 - T_{n+1}^{(2)} - (n - T_n^{(2)})| > \epsilon\} .$$

Conditioning on  $T_n^{(2)} = k$

$$\mathbb{P}(T_{n+1}^{(2)} = T_n^{(2)}) = \sum_{k=1}^{\infty} \mathbb{P}(T_{n+1}^{(2)} = k \mid T_n^{(2)} = k) \mathbb{P}(T_n^{(2)} = k) .$$

Since  $\nu$  has complete grammar, the above sum goes just up to  $n$ . Further, to get  $T_{n+1}^{(2)} = k$  whenever one has  $T_n^{(2)} = k$ , it is necessary and sufficient to have  $y_n = x_{n-k}$ , due also to the complete grammar. Thus,  $\mathbb{P}(T_{n+1}^{(2)} = k \mid T_n^{(2)} = k) = \sum_{y_n \in \chi} \mu\nu(y_n) = \mathbb{E}_{\mu,\nu}(1)$  which is positive since  $\mu$  is absolutely continuous respect to  $\nu$ . This finishes the proof.  $\square$

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